Maclaurin Olympiad 2018 Solutions

M1. The sum of the squares of two real numbers is equal to fifteen times their sum. The difference of the squares of the same two numbers is equal to three times their difference.

Find all possible pairs of numbers that satisfy the above criteria.

Solution

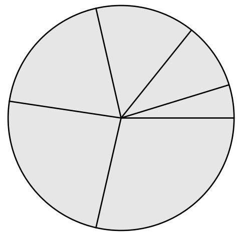
Let the numbers be a and b. Then $a^2 + b^2 = 15(a + b)$ and $a^2 - b^2 = \pm 3(a - b)$.

We do not know whether a and b are positive or negative, so if, for instance, $a = 4$ and $b = 3$, the expressions $a^2 - b^2$ and $a - b$ are both positive, but if $a = -4$ and $b = 3$ the first is positive and the second is negative. The ± sign is to take care of all alternatives. In fact, we have $a^2 + b^2 \geqslant 0$ in the first equation, so we know that $a + b \geqslant 0$. Now, by difference of two squares, we can write $a^2 - b^2$ as $(a - b)(a + b)$ in the second equation. Hence, if $a - b \neq 0$, we can cancel this term to obtain $a + b = \pm 3$, and so we know that the positive sign is appropriate.

We now have $a + b = 3$ and $a^2 + b^2 = 45$. Writing $b = 3 - a$, we obtain the quadratic $a^2 - 3a - 18 = 0$ and solutions $(6, -3)$ and $(-3, 6)$, so the numbers are 6 and -3 in either order.

However, there is also the possibility that $a - b = 0$, but if that is the case the first equation becomes $2a^2 = 30a$ and so $a = 0$ or 15. Hence the pair of numbers is either 0 and 0, 15 and 15 or 6 and -3.

M2. The diagram shows a circle that has been divided into six sectors of different sizes.

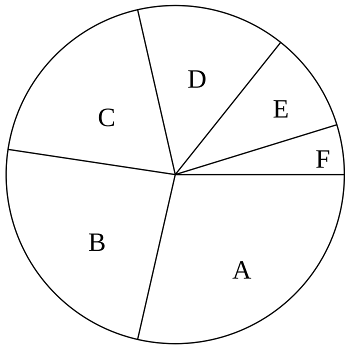


Two of the sectors are to be painted red, two of them are to be painted blue, and two of them are to be painted yellow. Any two sectors which share an edge are to be painted in different colours.

In how many ways can the circle be painted?

Solution

Label the sectors as in the diagram.



Suppose A is allocated a colour, which we call X. This can be done in three ways. Then neither B nor F can be coloured with X.

- If C is coloured with X, then B can be allocated a second colour in two ways. E must have the same colour as B since otherwise either D and E or E and F would have the same colour. Hence F and D have the third colour. This leads to 6 colourings.
- Similarly, if E is coloured with X, then F and C share a colour and B and D share the third. Again there are 6 colourings.
- If D is coloured with X, then E and F have different colours and, independently, so do B and C. This leads to a further 12 colourings.

The total number of colourings is 24.

M3. Three positive integers have sum 25 and product 360.
Find all possible triples of these integers.

Solution

Let the integers be a, b, c. Then $a + b + c = 25$ and $abc = 360 = 2^3 \times 3^2 \times 5$. We must split this factorised product up into parts so that the sum of the resulting numbers is 25.

It is possible, of course, to make a list of all the different ways of doing this, but there are 32 possibilities and ensuring that no alternatives are missing is difficult. Such an approach depends on the list being exhaustive and so omissions result in it being invalid. However, there are several methods which avoid making such a list.

Method 1:

This focuses on the fact that one of the three numbers is divisible by 5. Without loss of generality, take this to be a. It is either 5, 10, 15 or 20, since 25 is too large to obtain the required sum.

If $a = 5$, then $b + c = 20$ and $bc = 72$. It is possible to list all the factorisations of 72 and show that none of them produce a sum of 20. However, a better approach is to consider the quadratic $x^2 - 20x + 72 = 0$. The sum of its roots is 20 and the product is 72, so the roots will be b and c. However, the discriminant of this quadratic is negative, so there are no real roots.

This approach is useful in checking all such cases.

If $a = 10$, the quadratic is $x^2 - 15x + 36 = 0$, which has roots 3 and 12. Hence there is a solution 10, 3, 12.

If $a = 15$, the quadratic is $x^2 - 10x + 24 = 0$, which has roots 4 and 6, and so there is a solution 15, 4, 6.

If $a = 20$, the quadratic is $x^2 - 15x + 18 = 0$, and again the discriminant is negative and there are no roots.

Hence there are exactly two triples of numbers, namely 4, 6, 15 and 3, 10, 12.

Method 2:

This focuses on the largest number of the three. Since their sum is 25, this is greater than 8 and smaller than 24. Since it is also a factor of 360, the only possible values are 9, 10, 12, 15, 18 and 20. The approach in Method 1 can now be used to eliminate 9, 18 and 20. The values 10 and 12 result in the same triple.

Method 3:

This focuses on the 2s in the factorisation of 360. Since $a + b + c = 25$, they cannot be all even, nor can two be odd and one even, and since $abc = 360$, they cannot be all odd. It follows that two are even and one is odd. The factor of 8 is split between the two even numbers.

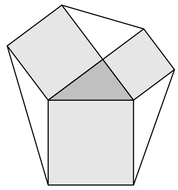
Without loss of generality, let $a = 2a_1$ and $b = 4b_1$, with a_1, b_1, c formed from the factors 3, 3 and 5. Now b_1 is not a multiple of 5 or 9, since the sum would be greater than 25, so it is either 1 or 3.

If $b_1 = 1$, then $b = 4$ and $2a_1 + c = 21$. Using the factors 3, 3 and 5, the only possibility is $a_1 = 3$, $c = 15$ and so $a = 6$.

If $b_1 = 3$, then $b = 12$ and $2a_1 + c = 13$. Using the factors 3 and 5, the only possibility is $a_1 = 5$, $c = 3$ and so $a = 10$.

Hence there are two triples of numbers, namely 4, 6, 15 and 3, 10, 12.

M4. The squares on each side of a right-angled scalene triangle are constructed and three further line segments drawn from the corners of the squares to create a hexagon, as shown. The squares on these three further line segments are then constructed (outside the hexagon).

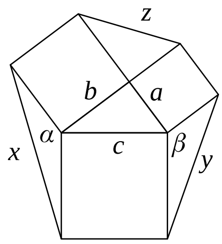


The combined area of the two equal-sized squares is 2018 cm^2.

What is the total area of the six squares?

Solution

Label the lengths as in the diagram below. Note that $z = c$ since it is the hypotenuse of a right-angled triangle with sides a and b. By chasing angles $\alpha = \pi - A$ and $\beta = \pi - B$. Hence $\cos\alpha = -\frac{a}{c}$ and $\cos\beta = -\frac{b}{c}$. We are told that $2c^2 = 2018$ so $c^2 = 1009$.



We have $x^2 = b^2 + c^2 - 2bc\cos\alpha = 3b^2 + c^2$ by the cosine rule, and similarly $y^2 = 3a^2 + c^2$.

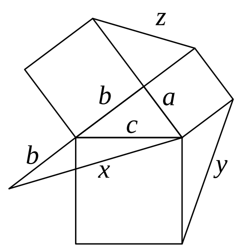
Now the sum of the six squares is

$$a^2 + b^2 + c^2 + (3b^2 + c^2) + (3a^2 + c^2) + c^2 = 4(a^2 + b^2) + 4c^2.$$

By Pythagoras, $a^2 + b^2 = c^2$ so the sum is $8c^2$, which is 8072.

Note that the cosine rule can be avoided by the construction shown. This produces a right-angled triangle whose sides are a and $2b$ whose hypotenuse is x.

It follows that $x^2 = a^2 + 4b^2$ and $y^2 = b^2 + 4a^2$.



M5. For which integers n is $\dfrac{16\left(n^2 - n - 1\right)^2}{2n - 1}$ also an integer?

Solution

We set $m = 2n - 1$, so $n = \frac{1}{2}(m + 1)$. Now the expression becomes

$$\begin{aligned}\frac{16}{m}\left[\left(\frac{m+1}{2}\right)^2 - \left(\frac{m+1}{2}\right) - 1\right]^2 &= \frac{1}{m}\left[(m+1)^2 - 2(m+1) - 4\right]^2 \\ &= \frac{1}{m}\left[m^2 - 5\right]^2 = \frac{m^4 - 10m^2 + 25}{m}.\end{aligned}$$

This is an integer if, and only if, m is a factor of 25. Hence the values of m are $\pm1,\ \pm5,\ \pm25$ and the corresponding values of n are $-12,\ -2,\ 0,\ 1,\ 3$ and 13.

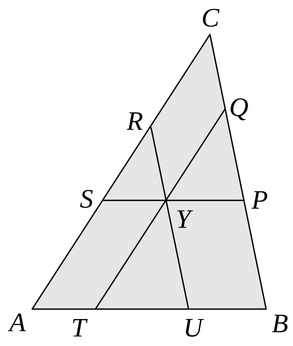
Instead of using the substitution, the numerator can be rewritten as $\left((2n - 1)^2 - 5\right)^2$ to obtain the same result.

M6. The diagram shows a triangle ABC and points T, U on the edge AB, points P, Q on BC, and R, S on CA, where:

(i) SP and AB are parallel, UR and BC are parallel, and QT and CA are parallel;

(ii) SP, UR and QT all pass through a point Y; and

(iii) $PQ = RS = TU$.

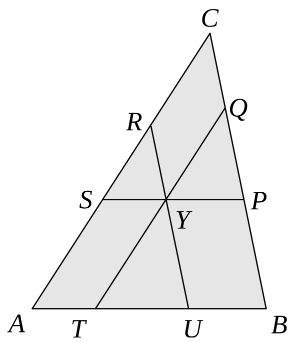


Prove that

$$\frac{1}{PQ} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}.$$

Solution

Let $TU = PQ = RS = k$, $AB = c$, $BC = a$ and $CA = b$.



Triangles $\triangle RSY$ and $\triangle CAB$ are similar, since RS is parallel to CA, SY is parallel to AB and YR is parallel to BC, so $SY = \dfrac{ck}{b}$.

Similarly $YP = \dfrac{ck}{a}$. Now

$$\begin{aligned} c &= AT + k + UB = SY + k + YP \\ &= \frac{ck}{b} + k + \frac{ck}{a} = \frac{ab + bc + ca}{ab}k \end{aligned}$$

and so $\dfrac{1}{c} = \dfrac{ab}{(ab + bc + ca)k}$.

If three such expressions are added, the factor $ab + bc + ca$ cancels and we obtain the desired result.